

# Spectral Problems of a Class of Non-self-adjoint One-dimensional Schrodinger Operators

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## Abstract

In this paper we investigate the one-dimensional Schrodinger operator  $L(q)$  with complex-valued periodic potential  $q$  when  $q \in L_1[0, 1]$  and  $q_n = 0$  for  $n = 0, -1, -2, \dots$ , where  $q_n$  are the Fourier coefficients of  $q$  with respect to the system  $\{e^{i2\pi nx}\}$ . We prove that the Bloch eigenvalues are  $(2\pi n + t)^2$  for  $n \in \mathbb{Z}$ ,  $t \in \mathbb{C}$  and find explicit formulas for the Bloch functions. Then we consider the inverse problem for this operator.

Key Words: Hill operator, Spectrum, Inverse problems.

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## 1 Introduction and Preliminary Facts

Let  $L(q)$  be the operator generated in  $L_2(-\infty, \infty)$  by the expression

$$-y''(x) + q(x)y(x) \quad (1)$$

with a complex-valued periodic potential  $q$ . In 1980, Gasymov [4] proved the following remarkable results for the operator  $L(q)$  with the potential  $q$  of the form

$$q(x) = \sum_{n=1}^{\infty} q_n e^{inx}, \quad (2)$$

where

$$\sum_n |q_n| < \infty.$$

**Result 1:** The spectrum  $S(L(q))$  of the operator  $L(q)$  is purely continuous and

$$S(L(q)) = [0, \infty). \quad (3)$$

There may be second order spectral singularity on the continuous spectrum which must coincide with numbers of the form  $(\frac{n}{2})^2$ .

**Result 2:** The equation

$$-y''(x) + q(x)y(x) = \mu^2 y(x) \quad (4)$$

has a solution of the form

$$f(x, \mu) = e^{i\mu x} \left( 1 + \sum_{n=1}^{\infty} \frac{1}{n + 2\mu} \sum_{\alpha=n}^{\infty} v_{n,\alpha} e^{i\alpha x} \right), \quad (5)$$

where the following series converge

$$\sum_{n=1}^{\infty} \frac{1}{n} \sum_{\alpha=n+1}^{\infty} \alpha(\alpha - n) |v_{n,\alpha}|, \quad \sum_{n=1}^{\infty} n |v_{n,\alpha}|.$$

**Result 3:** By the Floquet solutions (5) a spectral expansion was constructed.

**Result 4:** It was shown that the Wronskian of the Floquet solutions

$$f_n(x) =: \lim_{\mu \rightarrow \frac{n}{2}} (n - 2\mu) f(x, -\mu)$$

and  $f(x, \frac{n}{2})$  is equal to zero and therefore they are linearly dependent:

$$f_n(x) = s_n f(x, \frac{n}{2}). \quad (6)$$

It was proved that from the generalized norming numbers  $\{s_n\}$  one can effectively reconstruct  $\{q_n\}$ . That is, the inverse spectral problem was considered.

Guillemin and Uribe [5] investigated the boundary value problem (bvp) generated on  $[0, 2\pi]$  by (1) and the periodic boundary conditions when  $q \in Q_2^+$ , that is,  $q \in L_2[0, 2\pi]$  and has the form (2). It was proved that the eigenvalues of this bvp are  $n^2$  for  $n \in \mathbb{Z}$  and the corresponding root functions were studied. For the operator  $L(q)$  with the potential  $q \in Q_2^+$  the inverse spectral problem was investigated in detail by Pastur and Tkachenko [7] and the alternative proofs of (3) were provided by Shin [8], Carlson [1] and Christiansen [2].

In this paper, we first prove that if  $q \in L_1[0, 1]$ ,  $q(x+1) = q(x)$  and

$$q_n = 0, \quad \forall n = 0, -1, -2, \dots, \quad (7)$$

where  $q_n = (q(x), e^{i2\pi nx})$  and  $(\cdot, \cdot)$  is the inner product in  $L_2[0, 1]$ , then

$$S(L(q)) = [0, \infty), \quad S(L_t(q)) = \{(2\pi n + t)^2 : n \in \mathbb{Z}\} \quad (8)$$

for all  $t \in \mathbb{C}$ , where  $L_t(q)$  is the operator generated in  $L_2[0, 1]$  by (1) and the conditions

$$y(1) = e^{it} y(0), \quad y'(1) = e^{it} y'(0). \quad (9)$$

It is well-known that (see [3, 6]) the spectrum  $S(L(q))$  of the operator  $L(q)$  is the union of the spectra  $S(L_t(q))$  of the operators  $L_t(q)$  for  $t \in (-\pi, \pi]$ . Thus we prove (8) for more general case and as one can see from Theorem 1 that in a simple way. Moreover, we find explicit formulas for the Bloch functions and consider the inverse problem for this general case. The method of this paper is based on the following statements of our paper [9]:

*The large eigenvalues  $\lambda_n(t)$  and the corresponding normalized eigenfunctions  $\Psi_{n,t}(x)$  of the operator  $L_t(q)$  for  $q \in L_1[0, 1]$  and  $t \neq 0, \pi$ , satisfy the following asymptotic formulas*

$$\lambda_n(t) = (2\pi n + t)^2 + O\left(\frac{\ln |n|}{n}\right), \quad \Psi_{n,t}(x) = e^{i(2\pi n + t)x} + O\left(\frac{1}{n}\right). \quad (10)$$

*These asymptotic formulas are uniform with respect to  $t$  in  $[\rho, \pi - \rho]$ , where  $\rho \in (0, \frac{\pi}{2})$  (see Theorem 2 of [9]). Furthermore, the following formulas hold (see (22) and (28) in [9]):*

$$(\lambda_n(t) - (2\pi n + t)^2)(\Psi_{n,t}, e^{i(2\pi n + t)x}) = A_m(\lambda_n(t))(\Psi_{n,t}, e^{i(2\pi n + t)x}) + R_{m+1}(\lambda_n(t)), \quad (11)$$

where

$$A_m(\lambda) = \sum_{k=1,2,\dots,m} a_k(\lambda), \quad (12)$$

$$a_k(\lambda) = \sum_{n_1, n_2, \dots, n_k} \frac{q_{n_1} q_{n_2} \dots q_{n_k} q_{-n(k)}}{\prod_{s=1, 2, \dots, k} [\lambda - (2\pi(n - n(s)) + t)^2]}, \quad (13)$$

$$R_m(\lambda) = \sum_{n_1, n_2, \dots, n_m} \frac{q_{n_1} q_{n_2} \dots q_{n_m} (q \Psi_{n,t}, e^{i(2\pi(n - n(m)) + t)x})}{\prod_{s=1, 2, \dots, m} [\lambda - (2\pi(n - n(s)) + t)^2]} = O\left(\frac{\ln |n|}{n}\right)^m, \quad (14)$$

$$n(s) =: n_1 + n_2 + \dots + n_s \quad (15)$$

and the summations in (13) and (14) are taken with the conditions  $n(s) \neq 0$  for  $s = 1, 2, \dots$

## 2 On the Bloch Eigenvalues and Bloch Functions

Denote by  $L_1^+[0, 1]$  and  $L_1^-[0, 1]$  the set of all  $q \in L_1[0, 1]$  satisfying (7) and  $q_n = 0$  for  $n = 0, 1, 2, \dots$  respectively. The formula (11) immediately give us the following

**Theorem 1** *If  $q \in L_1^+[0, 1]$  then the eigenvalues of  $L_t(q)$  for  $t \in \mathbb{C}$  are  $(2\pi n + t)^2$ , where  $n \in \mathbb{Z}$ . These eigenvalues for  $t \neq \pi k$ , where  $k \in \mathbb{Z}$ , are simple. The eigenvalues  $(2\pi n)^2$  for  $n \in \mathbb{Z} \setminus \{0\}$  and  $(2\pi n + \pi)^2$  for  $n \in \mathbb{Z}$  are double eigenvalues of  $L_0(q)$  and  $L_\pi(q)$  respectively. The theorem continues to hold if  $L_1^+[0, 1]$  is replaced by  $L_1^-[0, 1]$ .*

**Proof.** Since at least one of the indices  $n_1, n_2, \dots, n_k, -n(k)$  (see (15)) is not positive number, by (7), (13) and (12)  $a_k(\lambda_n(t)) = 0$ ,  $A_m(\lambda_n(t)) = 0$  for all  $k, m$ . Therefore in (11) letting  $m$  tend to infinity and then using (14) and (10) we obtain  $\lambda_n(t) = (2\pi n + t)^2$  for  $t \in [\rho, \pi - \rho]$  and  $n > N(\rho) \gg 1$ . On the other hand  $\lambda_n(t)$  are the squares of the roots of

$$F(\mu) = 2 \cos t, \quad \forall t \in [\rho, \pi - \rho],$$

where  $F(\mu) = \varphi'(1, \mu) + \theta(1, \mu)$  and  $\varphi(x, \mu)$ ,  $\theta(x, \mu)$  are the solutions of the equation (4) satisfying the initial conditions  $\theta(0, \mu) = \varphi(0, \mu) = 1$ ,  $\theta'(0, \mu) = \varphi'(0, \mu) = 0$  (see [3]). Thus the entire functions  $F(\mu)$  and  $2 \cos \mu$  coincide on  $\{(2\pi n + t) : t \in [\rho, \pi - \rho]\}$ . Therefore these functions are identically equal in the complex plane and hence the eigenvalues of  $L_t(q)$  are the squares of the roots of the equation  $\cos \mu = \cos t$  for all  $t \in \mathbb{C}$ . That is, in the case  $q \in L_1^+[0, 1]$  the theorem is proved. The case  $q \in L_1^-[0, 1]$  can be proved in the same way ■

Now to consider the Bloch functions  $\Psi_{n,t}(x)$  corresponding to the eigenvalue  $(2\pi n + t)^2$  we use the equality

$$((2\pi n + t)^2 - (2\pi(n + p) + t)^2)(\Psi_{n,t}, e^{i(2\pi(n+p)+\bar{t})x}) = (q\Psi_{n,t}, e^{i(2\pi(n+p)+\bar{t})x}) \quad (16)$$

obtained from  $-\Psi_{n,t}'' + q\Psi_{n,t} = (2\pi n + t)^2 \Psi_{n,t}$  by multiplying  $e^{i(2\pi(n+p)+\bar{t})x}$ .

**Theorem 2** *Suppose  $q \in L_1^+[0, 1]$ . Let  $\Psi_{n,t}(x)$  be the eigenfunction of the operator  $L_t(q)$  corresponding to the eigenvalue  $(2\pi n + t)^2$  and normalized as*

$$(\Psi_{n,t}, e^{i(2\pi n + \bar{t})x}) = 1, \quad (17)$$

where  $t \neq \pi k$  for  $k \in \mathbb{Z}$ . Then

$$\Psi_{n,t}(x) = e^{i(2\pi n + t)x} + \sum_{p \in \mathbb{N}} c_{p,n}(t) e^{i(2\pi(n+p)+t)x}, \quad (18)$$

where

$$c_{p,n}(t) = d_{p,n}(t) \left( q_p + \sum_{k=1}^{p-1} \sum_{n_1, n_2, \dots, n_k} \frac{q_{n_1} q_{n_2} \dots q_{n_k} q_{p-n(k)}}{\prod_{s=1}^k (2\pi(2n+p-n(s)) + 2t) 2\pi(n(s)-p)} \right), \quad (19)$$

$d_{p,n}(t) = -(2\pi p(2\pi(2n+p) + 2t))^{-1}$  for  $p = 1, 2, \dots$  and

$$\{n_1, n_2, \dots, n_s, p - n_1 - n_2 - \dots - n_s\} \subset \mathbb{N} =: \{1, 2, \dots\} \quad (20)$$

for  $s = 1, 2, \dots, p-1$ . The theorem continues to hold if  $L_1^+[0, 1]$  is replaced by  $L_1^-[0, 1]$  and  $\mathbb{N}$  in (18) and (20) is replaced by  $-\mathbb{N}$ .

**Proof.** Let  $\Psi_{n,t}(x)$  be the normalized eigenfunction of the operator  $L_t(q)$  corresponding to the eigenvalue  $(2\pi n + t)^2$  and  $t \neq \pi k$  for  $k \in \mathbb{Z}$ . (In the end we prove that there exists an eigenfunction of the operator  $L_t(q)$  satisfying (17). For simplicity of notation we denote it also by  $\Psi_{n,t}$ ). Since the systems  $\{e^{i(2\pi n+t)x} : n \in \mathbb{Z}\}$  and  $\{e^{i(2\pi n+\bar{t})x} : n \in \mathbb{Z}\}$  are biorthogonal in  $L_2[0, 1]$  we have

$$\Psi_{n,t}(x) - (\Psi_{n,t}, e^{i(2\pi n+\bar{t})x}) e^{i(2\pi n+t)x} = \sum_{p \in \mathbb{Z} \setminus \{0\}} (\Psi_{n,t}, e^{i(2\pi(n+p)+\bar{t})x}) e^{i(2\pi(n+p)+t)x}. \quad (21)$$

To find  $(\Psi_{n,t}, e^{i(2\pi(n+p)+\bar{t})x})$  we iterate (16) by using

$$(q\Psi_{n,t}, e^{i(2\pi(n+p)+\bar{t})x}) = \sum_{n_1} q_{n_1} (\Psi_{n,t}(x), e^{i(2\pi(n+p-n_1)+\bar{t})x}) \quad (22)$$

(see (14) of [9]). Namely, using (22) and (16) we get

$$(\Psi_{n,t}, e^{i(2\pi(n+p)+\bar{t})x}) = d_{p,n}(t) \sum_{n_1} q_{n_1} (\Psi_{n,t}, e^{i(2\pi(n+p-n_1)+\bar{t})x}). \quad (23)$$

Now isolate the terms in the right-hand side of (23) containing the multiplicand

$(\Psi_{n,t}, e^{i(2\pi n+\bar{t})x})$  which occurs in the case  $n_1 = p$  and use (23) for the other terms to get

$$(\Psi_{n,t}, e^{i(2\pi(n+p)+\bar{t})x}) = d_{p,n}(t) (q_p (\Psi_{n,t}, e^{i(2\pi n+\bar{t})x}) + \sum_{n_1, n_2} \frac{q_{n_1} q_{n_2} (\Psi_{n,t}, e^{i(2\pi(n+p-n_1-n_2)+\bar{t})x})}{(2\pi(2n+p-n_1) + 2t) 2\pi(n_1-p)}).$$

Repeating this process  $m$  times, that is, isolating again the terms containing the multiplicand  $(\Psi_{n,t}, e^{i(2\pi n+\bar{t})x})$  which occurs in the case  $n_1 + n_2 = p$  and using again (23) for the other terms and doing this iteration  $m$  times, we obtain

$$(\Psi_{n,t}, e^{i(2\pi(n+p)+\bar{t})x}) = c_{p,n}(t) (\Psi_{n,t}, e^{i(2\pi n+\bar{t})x}) + r_m, \quad (24)$$

$$r_m = d_{p,n}(t) \sum_{n_1, n_2, \dots, n_m} \left( \frac{q_{n_1} q_{n_2} \dots q_{n_m} (q\Psi_{n,t}, e^{i(2\pi(n+p-n(m))+\bar{t})x})}{\prod_{s=1}^m [(2\pi(2n+p-n(s)) + 2t) 2\pi(n(s)-p)]} \right), \quad (25)$$

where  $m > p$ ,  $p - n(s) \neq 0$  for  $s = 1, 2, \dots, m$ .

Now we prove that  $r_m \rightarrow 0$  as  $m \rightarrow \infty$ . By (20),  $n_k \geq 1$  for  $k = 1, 2, \dots, m$  and hence  $n(s) \geq s$ . Using this and taking into account that  $(q\Psi_{n,t}, e^{i(2\pi(n+p-n(m))+\bar{t})x}) \rightarrow 0$  as

$m \rightarrow \infty$  from (25) we obtain

$$|r_m| \leq |d_{p,n}(t)| \prod_{s=1,2,\dots,m} \left( \sum_{j \geq s, j \neq p} \frac{M}{|(2\pi(2n+p-j)+2t)2\pi(j-p)|} \right) \quad (26)$$

for  $m \gg 1$ , where  $M = \sup_n |q_n|$ . Clearly, there exists  $K(t)$  such that

$$\sum_{j \geq s, j \neq p} \left| \frac{M}{(2\pi(2n+p-j)+2t)2\pi(j-p)} \right| \leq K(t). \quad (27)$$

for  $s = 1, 2, \dots, m$ . Moreover, if  $s > 4(|n| + |p|)$  then

$$\sum_{j \geq s} \left| \frac{M}{(2\pi(2n+p-j)+2t)2\pi(j-p)} \right| < \sum_{j \geq s} \frac{M}{j^2} < \frac{M}{s-1}. \quad (28)$$

Now using (26)-(28) we obtain

$$|r_m| \leq \frac{|d_{p,n}(t)| M^{m-4(|n|+|p|)} (K(t))^{4(|n|+|p|)}}{4(|n|+|p|)(4(|n|+|p|)+1)\dots(m-1)}$$

which implies that  $r_m \rightarrow 0$  as  $m \rightarrow \infty$ . Therefore in (24) letting  $m$  tend to infinity we get

$$(\Psi_{n,t}, e^{i(2\pi(n+p)+\bar{t})x}) = c_{p,n}(t)(\Psi_{n,t}, e^{i(2\pi n+\bar{t})x}). \quad (29)$$

This with (21) shows that  $(\Psi_{n,t}, e^{i(2\pi n+\bar{t})x}) \neq 0$ . Therefore, there exists eigenfunction, denoted again by  $\Psi_{n,t}$ , satisfying (17) and for this eigenfunction, by (29), we have

$$(\Psi_{n,t}, e^{i(2\pi(n+p)+\bar{t})x}) = c_{p,n}(t). \quad (30)$$

The indices  $n_1, n_2, \dots, n_k, p - n(k)$  taking part in the expression of  $c_{p,n}(t)$  (see (19)) satisfy (20). Therefore if  $p < 0$ , then the set of these indices is empty, that is, the first term on the right-hand side of (24) does not appear at all. Hence, from (24) using the relation  $r_m \rightarrow 0$  as  $m \rightarrow \infty$  we obtain

$$(\Psi_{n,t}, e^{i(2\pi(n+p)+\bar{t})x}) = 0, \quad \forall p < 0. \quad (31)$$

Thus (18) for  $q \in L_1^+[0, 1]$  follows from (21), (17), (30) and (31). The case  $q \in L_1^-[0, 1]$  can be considered in the same way. ■

### 3 On the Inverse Problem

First consider the Floquet solutions of the equation (4) defined by

$$\Psi(x, \mu) = \Psi_{n,t}(x) \quad (32)$$

for  $\mu = (2\pi n + t)$ , where  $n \in \mathbb{Z}$ ,  $\text{Re } t \in (-\pi, \pi]$  and  $\Psi_{n,t}(x)$  is studied in Theorem 2. Since  $\Psi_{n,t}(x)$  satisfies (9), we have

$$\Psi_{n,t}(x + m) = e^{itm} \Psi_{n,t}(x)$$

for all  $x \in (-\infty, \infty)$  and  $m \in \mathbb{Z}$ . This with the equality  $\text{Im } \mu = \text{Im } t$  implies that

$\Psi(x, \mu) \in L_2(a, \infty)$ ,  $\Psi(x, -\mu) \in L_2(-\infty, a)$  for  $\text{Im } \mu > 0$  and  $a \in (-\infty, \infty)$ . Therefore repeating the arguments of [4] one can obtain the spectral expansion. Note that we con-

struct the Floquet solution for more general case and by the other method (see Result 2 in introduction).

Now we consider the inverse problem as follows. We write the Fourier decomposition of  $\Psi_{n,t}(x)$  and  $\Psi_{-n,-t}(x)$  in the form

$$\Psi_{n,t}(x) = \sum_{p \in \mathbb{Z}} c_{p,n}(t) e^{i(2\pi(n+p)+t)x}, \quad \Psi_{-n,-t}(x) = \sum_{p \in \mathbb{Z}} c_{p,-n}(-t) e^{i(2\pi(-n+p)-t)x}, \quad (33)$$

where, by Theorem 2,  $c_{p,n}(t)$  for  $p > 0$  is defined by (19) and

$$c_{0,n}(t) = 1, \quad c_{p,n}(t) = 0, \quad \forall p < 0. \quad (34)$$

First we show that

$$\lim_{t \rightarrow 0} 8n\pi c_{2n+p,-n}(-t) = c_{p,n}(0) s_{2n}, \quad \forall p \geq -2n, \quad (35)$$

$$\lim_{t \rightarrow \pi} 4(2n+1)(t-\pi)\pi c_{2n+p+1,-n}(-t) = c_{p,n}(\pi) s_{2n+1}, \quad \forall p \geq -2n-1, \quad (36)$$

where

$$s_n = q_n + \sum_{k=1}^{n-1} S_k(n), \quad S_k(n) = \sum_{n_1, n_2, \dots, n_k} \frac{q_{n_1} q_{n_2} \dots q_{n_k} q_{n-n(k)}}{\prod_{s=1}^k (2\pi n(s)) 2\pi(n-n(s))}, \quad n = 1, 2, \dots \quad (37)$$

(see Lemma 1). Then using these equalities we prove that

$$\lim_{t \rightarrow 0} 8n\pi t \Psi_{-n,-t}(x) = s_{2n} \Psi_{n,0}(x), \quad \forall n \geq 1, \quad (38)$$

$$\lim_{t \rightarrow \pi} 4(2n+1)(t-\pi)\pi \Psi_{-n,-t}(x) = s_{2n+1} \Psi_{n,\pi}(x), \quad \forall n \geq 0 \quad (39)$$

(see Lemma 2 and Theorem 3). Due to (32) and (6)  $\{s_n\}$  is the sequence of the norming numbers. Finally, we investigate the property of the norming numbers and consider the question when  $\{s_n\}$  may be a sequence of the norming number for the operator  $L(q)$  with potential  $q \in L_1^+[0, 1]$  (see Theorem 4, Proposition 1 and Corollary 1).

**Lemma 1** *The equalities (35) and (36) hold for all  $n \geq 1$  and  $n \geq 0$  respectively.*

**Proof.** The proof of (35) for  $p = -2n$  follows from (34). From the definition of  $d_{p,n}(t)$  (see Theorem 2) we see that

$$d_{2n+p,-n}(0) = d_{p,n}(0), \quad \lim_{t \rightarrow 0} 8n\pi t d_{2n,-n}(-t) = 1, \quad (40)$$

$$\lim_{t \rightarrow 0} 8n\pi t d_{2n+p,-n}(-t) = 0, \quad \forall p \neq 0, -2n. \quad (41)$$

Therefore by (19) and (37) we have

$$\lim_{t \rightarrow 0} 8n\pi t c_{2n,-n}(-t) = s_{2n}.$$

Thus the proof of (35) for  $p = 0$  also follows from (34).

To prove (35) in the more complicated cases  $p \neq 0, -2n$  we rewrite  $c_{2n+p,-n}(-t)$ ,  $c_{p,n}(0)$

and  $s_{2n}$  in the following form

$$c_{2n+p,-n}(-t) = \sum_{k=0}^{2n+p-1} D_k(-t), \quad D_0(-t) = d_{2n+p,-n}(-t)q_{2n+p}, \quad (42)$$

$$D_k(-t) = \sum_{n_1, n_2, \dots, n_k} D_k(-t, n_1, n_2, \dots, n_k), \quad \forall k > 0, \quad (43)$$

$$D_k(-t, n_1, n_2, \dots, n_k) = \frac{d_{2n+p,-n}(-t)q_{n_1}q_{n_2}\dots q_{n_k}q_{2n+p-n(k)}}{\prod_{s=1}^k (2\pi(p-n(s)) - 2t)2\pi(n(s) - 2n - p)}, \quad (44)$$

$$c_{p,n}(0) = \sum_{j=0}^{p-1} F_j, \quad F_0 = d_{p,n}(0)q_p, \quad (45)$$

$$F_j = d_{p,n}(0) \sum_{n_1, n_2, \dots, n_j} \frac{q_{n_1}q_{n_2}\dots q_{n_j}q_{p-n(j)}}{\prod_{s=1}^j (2\pi(p-n(s)))2\pi(n(s) - 2n - p)}, \quad (46)$$

$$s_{2n} = \sum_{i=0}^{2n-1} E_i, \quad E_0 = q_{2n}, \quad E_i = \sum_{m_1, m_2, \dots, m_i} E(m_1, m_2, \dots, m_i), \quad (47)$$

$$E(m_1, m_2, \dots, m_i) = \frac{q_{m_1}q_{m_2}\dots q_{m_i}q_{2n-m(i)}}{\prod_{s=1}^i (2\pi m(s))2\pi(2n - m(s))}. \quad (48)$$

By (41)

$$\lim_{t \rightarrow 0} 8n\pi t D_0(-t) = 0. \quad (49)$$

Now let us investigate  $D_k(-t)$  for  $1 \leq k \leq 2n+p-1$  and  $p \neq 0, -2n$ . By (44), (7) and by (15) and (20) we have  $2n+p-n(k) > 0$  and  $n(s) < n(k)$  for  $s < k$ . Therefore the multiplicand  $n(s) - 2n - p$  of the denominator of the fraction in (44) is a negative integer:

$$n(s) - 2n - p < 0 \quad (50)$$

for  $s = 1, 2, \dots, k$ . To investigate the other multiplicand  $p - n(s)$  consider the cases:

Case 1:  $-2n < p < 0$ . Since  $n(s) > 0$ , we have  $p - n(s) \neq 0$ . This with (50) gives

$$\lim_{t \rightarrow 0} 8n\pi t D_k(-t) = 0.$$

Therefore (35) follows from (34), (42) and (49).

Case 2:  $p > 0$ . One can readily see that  $D_k(-t)$  can be written in the form

$$D_k(-t) = \sum_{j=-1}^{p-1} D_{k,j}(-t), \quad (51)$$

where  $D_{k,-1}(-t)$  and  $D_{k,j}(-t)$  for  $j \geq 0$  are the right-hand side of (43) when the summation is taken under conditions

$$p - n(s) \neq 0, \quad \forall s = 1, 2, \dots, k \quad (52)$$

and

$$n_1 + n_2 + \dots + n_{j+1} = p. \quad (53)$$

respectively. By (52), (50) and (44) we have

$$\lim_{t \rightarrow 0} 8n\pi t D_{k,-1}(-t) = 0. \quad (54)$$

Now consider  $D_{k,j}(-t)$  for  $j \geq 0$ , i.e., assume that (53) holds. The indices  $n_1, n_2, \dots, n_{j+1}$  satisfying (53) take part in  $D_{k,j}(-t)$  if and only if  $j+1 \leq k \leq 2n+j$ , since  $n_s > 0$  for all  $s = 1, 2, \dots$  and  $2n+p-n(k) > 0$  (see (44)). Therefore

$$D_{k,j}(-t) = 0 \quad (55)$$

for  $k \leq j$  and for  $k > 2n+j$ . Thus it remains to consider  $D_{k,j}(-t)$  for  $j \geq 0$  and  $j+1 \leq k \leq 2n+j$ . If (53) holds then  $n_{j+1} = p - n(j)$  and by (44) the expression  $D_k(-t, n_1, n_2, \dots, n_k)$  can be written as product of

$$\frac{d_{2n+p,-n}(-t) q_{n_1} q_{n_2} \dots q_{n_j} q_{p-n(j)}}{\left( \prod_{s=1}^j (2\pi(p-n(s)) - 2t) 2\pi(n(s) - 2n-p) \right) 8n\pi t}$$

and

$$\frac{q_{n_{j+2}} q_{n_{j+3}} \dots q_{n_k} q_{2n-n_{j+2}-n_{j+3}-\dots-n_k}}{\prod_{s=j+2}^k (2\pi(n_{k+2} + n_{k+3} + \dots + n_s)) 2\pi(2n - n_{k+2} - n_{k+3} - \dots - n_s)}.$$

The last expression is  $E(m_1, m_2, \dots, m_{k-j-1})$  for  $m_1 = n_{j+2}$ ,  $m_2 = n_{j+3}$ ,  $\dots$ ,  $m_{k-j-1} = n_k$  (see (48)). Using this and (40) and taking into account that  $p - n(s) \neq 0$  for all  $s \neq j+1$  (see (53) and use the inequality  $n_s > 0$  for all  $s = 1, 2, \dots$ ) we obtain

$$\lim_{t \rightarrow 0} 8n\pi t D_{k,j}(-t) = F_j \sum_{m_1, m_2, \dots, m_{k-j-1}} E(m_1, m_2, \dots, m_{k-j-1}).$$

This with (55), (47) and (45) implies that

$$\lim_{t \rightarrow 0} 8n\pi t \sum_{k=0}^{2n+p-1} D_{k,j}(-t) = F_j \sum_{k=j+1}^{2n+j} E_{k-j-1} = F_j s_{2n},$$

$$\lim_{t \rightarrow 0} 8n\pi t \sum_{j=0}^{p-1} \sum_{k=0}^{2n+p-1} D_{k,j} = s_{2n} \sum_{j=0}^{p-1} F_j = s_{2n} c_{p,n}(0).$$

Thus (35) follows from (42), (51) and (54). In the same way one can prove (36) ■

**Lemma 2** For any  $n \geq 1$  and  $n \geq 0$  there exist constants  $K$  and  $L$  such that the inequalities

$$8n\pi t | (q\Psi_{-n,-t}, e^{i(2\pi m-t)x}) | < K, \quad \forall m \in \mathbb{Z} \quad (56)$$

and

$$4(2n+1)\pi(\pi-t) | (q\Psi_{-n,-t}, e^{i(2\pi m-t)x}) | < L, \quad \forall m \in \mathbb{Z} \quad (57)$$

hold for  $t \in (0, \frac{\pi}{2})$  and for  $t \in [\frac{\pi}{2}, \pi)$  respectively.

**Proof.** Let us prove (56) for  $t \in (0, \frac{\pi}{2})$ . Since  $(q(x)\Psi_{-n,-t}(x), e^{i(2\pi m-t)x})$  tends to zero as  $|m| \rightarrow \infty$ , there exists a constant  $C(t)$  and integer  $k_0(t)$  such that

$$\max_{m \in \mathbb{Z}} | (q(x)\Psi_{-n,-t}(x), e^{i(2\pi m-t)x}) | = | (q(x)\Psi_{-n,-t}(x), e^{i(2\pi k_0-t)x}) | = C(t). \quad (58)$$



Let  $l$  be an integer such that

$$\sum_{k \geq l} \frac{1}{k^2} < \frac{1}{2M}, \quad (59)$$

where  $M$  is defined in (26). Using (58), (22) and then (30), (59) we obtain

$$\begin{aligned} C(t) &= \left| (q\Psi_{-n,-t}, e^{i(2\pi k_0 - t)x}) \right| \leq \left| \sum_{m: |k_0 - m| \leq |n| + l} q_m(\Psi_{-n,-t}, e^{i(2\pi(k_0 - m) - t)x}) \right| \\ &+ \left| \sum_{m: |k_0 - m| > |n| + l} \frac{q_m(q(x)\Psi_{N,t}(x), e^{i(2\pi(k_0 - m) + t)x})}{(-2\pi n - t)^2 - (2\pi(k_0 - m) - t)^2} \right| < S + \frac{C(t)}{2}, \end{aligned} \quad (60)$$

where

$$S = M \sum_{p: |p| \leq 2|n| + l} |c_{-n,p}(-t)|. \quad (61)$$

On the other hand, from (19) and (20) one can readily see that there exists a constant  $c$  such that  $8n\pi t |c_{-n,p}(-t)| < c$  for all  $p$  with  $|p| \leq 2|n| + l$ . Moreover the number of the summands in (61) is less than  $2(2|n| + l + 1)$ . Therefore  $8n\pi t |S| < 2M(2|n| + l + 1)c$ . This inequality with (60) implies that

$$8n\pi t C(t) < 2M(2|n| + l + 1)c + \frac{8n\pi t C(t)}{2},$$

that is, (56) holds for  $K = 4M(2|n| + l + 1)c$ . In the same way we prove (57). ■

**Theorem 3** *If  $q \in L_1^+[0, 1]$  then (38) and (39) hold for  $n \geq 1$  and  $n \geq 0$  respectively.*

**Proof.** From (30), (16) and Lemma 2 it follows that

$$8n\pi t |c_{p,-n}(-t)| = |8n\pi t d_{p,-n}(-t)(q\Psi_{-n,-t}, e^{i(2\pi(-n+p)-t)x})| < K |d_{p,-n}(t)| \quad (62)$$

for  $t \in (0, \frac{\pi}{2})$ . Therefore the series

$$8n\pi t \Psi_{-n,-t}(x) = 8n\pi t e^{i(-2\pi n - t)x} + \sum_{p \in \mathbb{N}} 8n\pi t c_{p,-n}(-t) e^{i(2\pi(-n+p)-t)x} \quad (63)$$

(see (18)) converges uniformly with respect to  $x \in [0, 1]$  and  $t \in (0, \frac{\pi}{2}]$ . Thus, in (63) letting  $t$  tend to zero and using equality (35) we get the proof of (38). To prove (39) instead of (35) and (56) we use (36) and (57) and repeat the proof of (38). ■

From (38) and (39) we define the norming numbers  $s_n$  for  $n = 1, 2, \dots$  By (37)

$$s_1 = q_1, \quad s_2 = q_2 + \frac{q_1^2}{(2\pi)^2}, \quad s_3 = q_3 + \frac{q_1 q_2}{(2\pi)^2} + \frac{q_1^3}{4(2\pi)^4}, \dots \quad (64)$$

Thus if the norming numbers  $s_n$  for  $n = 1, 2, \dots$  are given then one can define recurrently

$$q_1 = s_1, \quad q_2 = s_2 - \frac{s_1^2}{(2\pi)^2}, \quad q_3 = s_3 - \frac{s_1 s_2}{(2\pi)^2} + \frac{3s_1^3}{4(2\pi)^4}, \dots \quad (65)$$

Now we are ready to prove the main result of this section. Let  $S$  be the set of all sequences  $\{s_n\}$  for which there exists  $s \in L_1[0, 1]$  with  $(s(x), e^{i2\pi n x}) = s_n$  for all  $n = 1, 2, \dots$

**Theorem 4** *For every  $q \in L_1^+[0, 1]$  the sequence  $\{s_n\}$  of norming numbers is an element of  $S$ . Conversely, for any sequence  $\{s_n\}$  from  $S$  there exists unique  $q \in L_1^+[0, 1]$  such that*

the sequence of the norming numbers of  $L(q)$  coincides with  $\{s_n\}$  if and only if the solutions  $q_1, q_2, \dots$  of (37) is a bounded sequence.

**Proof.** Since  $|q_n| \leq M$  for all  $n$ , where  $M$  is defined in (26), from (37) we obtain

$$|S_1(n)| \leq \frac{M^2}{(2\pi)^2} \sum_{k=1}^{n-1} \frac{1}{k(n-k)}. \quad (66)$$

In the same way we get

$$|S_k(n)| \leq \frac{M^{k+1}}{(2\pi)^{2k}} \left( \sum_{k=1}^{n-1} \frac{1}{k(n-k)} \right)^k. \quad (67)$$

Now using the obvious inequality

$$\sum_{k=1}^{n-1} \frac{1}{k(n-k)} < \frac{2(1 + \ln n)}{n}, \quad \forall n \geq 4 \quad (68)$$

from (37) we obtain that

$$s_n - q_n = O(n^{-1} \ln n). \quad (69)$$

Therefore there exists  $p \in L_2[0, 1]$  such that  $(p(x), e^{i2\pi nx}) = s_n - q_n$  for all  $n = 1, 2, \dots$ . Then the function  $s(x) = p(x) + q(x)$  belongs to  $L_1[0, 1]$  and  $(s(x), e^{i2\pi nx}) = s_n$  for all  $n = 1, 2, \dots$ , that is,  $\{s_n\} \in S$ .

Now suppose that  $\{s_n\} \in S$  and the solutions  $q_1, q_2, \dots$  of (37) is a bounded sequence. Then there exists a constant  $C$  such that  $|q_n| \leq C$  for  $n = 1, 2, \dots$ . Instead of  $M$  using  $C$  and repeating the proof of (69) we see that  $\{q_n\} \in S$ . Therefore there exists unique  $q \in L_1^+[0, 1]$  such that the sequence of the norming numbers of  $L(q)$  coincides with  $\{s_n\}$ . ■

It remains to find the conditions on the sequence  $\{s_n\}$  of norming numbers such that the sequence  $\{q_n\}$  defined from (37) is bounded. Below we present an example by using the following obvious inequality

$$\sum_{k=1}^{n-1} \frac{1}{k(n-k)} \leq 1, \quad \forall n > 1. \quad (70)$$

which follows from (68) for  $n > 5$  and can be verified by calculations for  $n \leq 5$ .

**Proposition 1** *If the sequence  $\{s_n\}$  of norming numbers satisfies the inequality*

$$|s_n| \leq 2\pi - \frac{2\pi}{2\pi - 1}, \quad \forall n = 1, 2, \dots \quad (71)$$

*then for the sequence  $\{q_n\}$  defined from (37) the following estimations hold*

$$|q_m| \leq 2\pi, \quad \forall m = 1, 2, \dots \quad (72)$$

**Proof.** Let us prove (72) by induction. It follows from (65) that (72) holds for  $m = 1, 2$ . Assume that (72) holds for  $m < n$ , where  $n > 2$ . Then  $|q_m| \leq 2\pi$  for  $m < n$ . The indices  $n_1, n_2, \dots, n_k, n - n_k$  taking part in the expressions of  $S_k(n)$  for  $k = 1, 2, \dots, n - 1$  (see (37)) less than  $n$ , since they are positive numbers and their total sum is  $n$ . Therefore by assumption of the induction the Fourier coefficients taking part in those expressions satisfy (72). Thus

in (67) instead of  $M$  taking  $2\pi$  and then using (70) we get

$$\sum_{k=1}^{n-1} |S_k(n)| \leq \sum_{k=1}^{n-1} \frac{(2\pi)^{k+1}}{(2\pi)^{2k}} < \frac{2\pi}{2\pi-1}.$$

This with (37) and (71) implies that  $|q_n| \leq |s_n| + \frac{2\pi}{2\pi-1} \leq 2\pi$  ■

Theorem 4 with Proposition 1 implies

**Corollary 1** *For any sequence  $\{s_n\} \in S$  satisfying (71) there exists unique  $q \in L_1^+[0,1]$  such that the sequence of the norming numbers of  $L(q)$  coincides with  $\{s_n\}$ .*

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